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LINEAR ISOMETRY ON REFLEXIVE STRONGLY FACIALLY SYMMETRIC SPACES

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Absrtact: In this paper, the isometry properties of SFS-spaces are studied. It is proved that a linear operator is a surjective isometry if and only if it maps the set of indecomposable geometric tripotents onto itself and preserves the orthogonality relations on the set of indecomposable geometric tripotents.

Keywords: strongly facially symmetric space, norm exposed face, geometric tripotent, geometric Peirce projections, surjective isometry.

In 1989, the paper [1] of Friedman and Russo was published, in which facially symmetric spaces were introduced; the main purpose for introducing these spaces was a geometric characterization of the predual spaces of JB*-triples admitting an algebraic structure. Many properties required in these characterizations are natural assumptions for state spaces of physical systems. These spaces are considered as a geometric model for states in quantum mechanics. It was proved that the predual space of a complex von Neumann algebra and more general of a JB*-triples is a neutral strongly facially symmetric space (see [2]). Neal and Russo [3] found geometric conditions under which a facially symmetric space is isometric to the predual space of a complex JBW* triple. In [4], a complete description of strongly facially symmetric spaces that are isometrically isomorphic to the predual space of an atomic commutative von Neumann algebra was obtained. In [5], it was proved that the predual space of a JBW-algebra is a strongly facially symmetric space if and only if the algebra is the direct sum of an Abelian algebra and an algebra of type I_2.

In this paper, the isometry properties of SFS-spaces are studied. It is proved that a linear operator is a surjective isometry if and only if it maps the set of indecomposable geometric tripotents onto itself and preserves the orthogonality relations on the set of indecomposable geometric tripotents.

Let Z be a real or complex normed space. We say that elements $x, y \quad Z$ are orthogonal and write $f \quad g$ if $\|f+g\| = \|f-g\| = \|f\| + \|g\|$. A face F of the unit ball $Z_1 = \{f \quad Z : \|f\| \quad 1\}$ is said to be norm exposed if $F = F_u = \{f \quad Z_1 : u(f) = 1\}$ for some $u \quad Z^*$ with $\|u\| = 1$. An element $u \quad Z^*$ is called a projective unit if $\|u\| = 1$ and $u(g) = 0$ for all $g \quad F_u$ (see [1]).

A norm exposed face F_u in Z is called a symmetric face if there exists a linear isometry S_u from Z to Z such that $S_u^2 = I$ whose fixed point set coincides with the topological direct sum of the closure $\overline{sp}\, F_u$ of the linear hull of the face F_u and its orthogonal complement F_u, i.e., with $\left(\overline{sp}\, F_u\right) \quad F_u$.

A space Z is said to be weakly facially symmetric (WFS) if each norm exposed face in Z_1 is symmetric.

For each symmetric face F_u, contractive projections $P_k(u)$, $k = 0,1,2$, on Z are defined as follows. First, $P_1(u) = (I - S_u)/2$ is the projection onto the eigenspace corresponding to the eigenvalue -1 of the symmetry S_u. Next, $P_2(u)$ and $P_0(u)$ are defined as projections of Z onto $\overline{sp}\,F_u$ and F_u, respectively. The projections $P_k(u)$ are called the geometric Peirce projections.

A WFS space Z is said to be strongly facially symmetric (SFS) if, for each norm exposed face F_u of Z and each y Z^* satisfying the conditions $\|y\|=1$ and F_u F_y, we have $S_u^* y = y$, where S_u is the symmetry corresponding to F_u.

A projective unit u from Z^* is called a geometric tripotent if F_u is a symmetric face and $S_u^* u = u$ for the symmetry S_u corresponding to F_u. In [6, 7], a necessary and sufficient condition was found when an element of the dual space of a reflexive SFS space is geometric tripotent.

Geometric tripotents u and v are said to be orthogonal if u $P_0(v)^* Z^*$ (which implies v $P_0(u)^* Z^*$) or, equivalently, $u \pm v$ GU.

Lemma 1. Let Z be a SFS-space and $\Phi : Z \to Z$ be a linear isometry. If u, v GU and u v, then $\Phi^* u$ $\Phi^* v$.

The set of geometric tripotents is ordered as follows: given u, v GU, we set u v if F_u F_v. A geometric tripotent u is called indecomposable if for v GU from v u, it follows $v = u$. By I we denote the set of all indecomposable geometric tripotents.

Lemma 2. Let Z be a SFS-space and $\Phi : Z \to Z$ be a surjective linear isometry. Then $\Phi^* u$ I for all u I.

Theorem 3. Let Z be an reflexive strongly facially symmetric space and $\Phi : Z \to Z$ be a linear operator. Then the following statements are equivalent:

1) Φ^* is a surjective isometry;
2) Φ^* maps I onto itself and preserves orthogonality relations on I.

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