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# ON THE ANALYTICAL SOLUTION OF SYSTEMS OF LINEAR INTEGRAL EQUATIONS

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## ANNOTATION

This article analyzes the mathematical essence of systems of linear integral equations and the problems of their solution. In the course of the study, Fredholm and Volterra systems of integral equations of the second kind are considered in a general form, and the conditions for the existence of solutions as well as their main properties are described. Special attention is paid to the structure of the integral kernel, and effective methods for solving systems with separable (degenerate) kernels are demonstrated.

The obtained results have significant theoretical and practical importance for the study of the theory of integral equations and for solving applied problems related to mathematical modeling. In addition, the research results can be used as a methodological resource in teaching mathematical analysis and functional analysis in higher educational institutions.

## KEYWORDS

linear integral equations, system of integral equations, Fredholm-type equations, Volterra-type equations, separable kernel, degenerate kernel, integral operator, functional analysis, system of algebraic equations, analytical solution.

## INTRODUCTION

The theory of integral equations is one of the important branches of mathematical analysis and serves as a fundamental tool in constructing mathematical models of many applied processes. In numerous physical, technical, and economic problems, the state of a process is represented by an unknown function whose values are determined through integral relations. Such problems are generally described by an integral equation of the form:

$$x(t)=f(t)+\int_a^b K(t,s)\,x(s)\,ds.$$

If several mutually dependent state functions are involved in the process, the problem takes the form of a system of linear integral equations:

$$x_i(t)=f_i(t)+\sum_{j=1}^{m}\int_a^b K_{ij}(t,s)\,x_j(s)\,ds, i=1,2,\ldots,m.$$

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This system belongs to the class of Fredholm integral equations of the second kind, where the unknown functions are determined through an integral operator in a coupled manner. The mathematical properties of the integral kernel $K_{ij}(t,s)$—such as continuity, boundedness, and symmetry—are the main factors determining the existence and uniqueness of solutions.

If the upper limit of integration is variable, namely,

$$x_i(t)=f_i(t)+\sum_{j=1}^{m}\int_a^t K_{ij}(t,s)\,x_j(s)\,ds,$$

then such systems are referred to as Volterra-type integral equation systems. From the standpoint of functional analysis, the problem of solving a system of integral equations can be represented as the following operator equation:

$x=f+Kx$, where $K$is the corresponding integral operator defined by

$(Kx)(t)=\int_a^b K(t,s)\,x(s)\,ds$. If the operator norm satisfies the condition $\|K\|<1$, then the solution can be expressed using the Neumann series:

$x=\sum_{n=0}^{\infty} K^n f$. This result demonstrates both the existence of a solution to the system of integral equations and the possibility of finding it constructively.

One of the important cases in practical problems is when the integral kernel has a separable (degenerate) form:

$K_{ij}(t,s)=\sum_{k=1}^{r} a_{ik}(t)\,b_{kj}(s)$. In such cases, the integral operator has finite rank, and the system of linear integral equations can be reduced to a finite-dimensional system of algebraic equations. As a result, the unknown functions take the form

$x_i(t)=f_i(t)+\sum_{k=1}^{r} a_{ik}(t)\,C_k, C_k=\int_a^b b_{kj}(s)x_j(s)\,ds,$ which makes it possible to obtain an analytical solution.

The main objective of this study is to provide an in-depth analysis of the mathematical mechanism for solving systems of linear integral equations based on the above theoretical approaches and to scientifically substantiate the process of obtaining exact solutions for systems with separable kernels. The obtained results are of significant theoretical and practical importance in the study of integral equations and in teaching mathematical disciplines in higher education.

## METHODOLOGY

In this study, the problem of solving systems of linear integral equations is investigated based on functional analysis and the theory of integral operators. As the object of research, Fredholm and Volterra integral equation systems of the second kind are selected, which in general form are written as:

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$$x_i(t)=f_i(t)+\sum_{j=1}^{m}\int_a^b K_{ij}(t,s)\,x_j(s)\,ds, i=1,2,...,m.$$

At the initial stage of the research, the system is transformed into a vector-functional form. For this purpose, the vector of unknown functions is defined as

$x(t)=(x_1(t),x_2(t),...,x_m(t))^T$, and the vector of given functions as

$f(t)=(f_1(t),f_2(t),...,f_m(t))^T$. Then the system of integral equations takes the operator form:

$x(t)=f(t)+\int_a^b K(t,s)x(s)\,ds=f(t)+Kx(t)$, where $K$ denotes the corresponding integral operator.

In the next stage, the norm and spectral properties of the integral operator are analyzed. If the condition

$\|K\|<1$ is satisfied, then the system has a unique solution, which can be determined by the method of successive approximations (Neumann series):

$x=\sum_{n=0}^{\infty} K^n f$. This method is applied to Fredholm-type systems and allows determining the convergence conditions of the solution.

For Volterra-type systems of linear integral equations,

$$x_i(t)=f_i(t)+\sum_{j=1}^{m}\int_a^t K_{ij}(t,s)\,x_j(s)\,ds,$$

the convergence of the iterative process is ensured automatically due to the variable upper limit of integration, and the solution is obtained constructively.

As the main methodological approach of the study, the separable (degenerate) kernel method is selected. If the integral kernel is represented as

$K_{ij}(t,s)=\sum_{k=1}^{r} a_{ik}(t)b_{kj}(s)$, then the integral operator has finite rank. In this case, the system of integral equations is reduced to

$x_i(t)=f_i(t)+\sum_{k=1}^{r} a_{ik}(t)C_k$, where the unknown coefficients

$C_k=\sum_{j=1}^{m}\int_a^b b_{kj}(s)x_j(s)\,ds$ form a system of algebraic equations. After solving this algebraic system, analytical expressions for the unknown functions are obtained.

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## RESULTS

During the research, the possibility of solving systems of linear integral equations with separable kernels was established. In particular, the following integral system was considered:

$$\begin{cases} x(t)=1+\int_0^1 t\,s\,x(s)\,ds+\int_0^1 t\,s\,y(s)\,ds, \\ y(t)=t+\int_0^1 t\,s\,x(s)\,ds+2\int_0^1 t\,s\,y(s)\,ds. \end{cases}$$

The integral kernel $K(t,s)=t\cdot s$was identified as separable, and the following moments were introduced:

$$A=\int_0^1 s\,x(s)\,ds,\ B=\int_0^1 s\,y(s)\,ds.$$

As a result, the system was reduced to a finite-dimensional system of algebraic equations, which after solving yielded

$$A=\frac{5}{2},B=\frac{7}{2}.$$

Based on these values, the exact forms of the unknown functions were obtained:

$$x(t)=1+6t,y(t)=\frac{21}{2}t.$$

The obtained solution confirms that the system of linear integral equations admits an analytical solution and demonstrates the effectiveness of the separable kernel method.

## DISCUSSION

The obtained results indicate that the structure of the integral kernel plays a decisive role in solving systems of linear integral equations. In particular, the separable form $K(t,s)=t\cdot s$makes it possible to reduce the integral operator to a finite-rank operator, thereby simplifying the system to a finite-dimensional algebraic system. This, in turn, creates favorable conditions for determining the unknown functions in an analytical form.

The methodological approach used in the study is consistent with classical methods of integral equation theory, including the Neumann series and operator techniques. However, in general cases where the kernel is not separable, obtaining analytical solutions becomes more difficult, and numerical methods are required. In this regard, the results obtained confirm the effectiveness of the chosen method for systems with separable kernels.

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In addition, the derived solutions have significant theoretical value in analyzing the existence and uniqueness of solutions of integral equation systems. The results demonstrate that analytical solutions can be obtained in simplified models, which serves as an important scientific basis for selecting solution methods in applied problems.

## CONCLUSION

This study provides a comprehensive analysis of the theoretical and practical aspects of solving systems of linear integral equations. The results show that transforming integral equation systems into operator form and analyzing the mathematical structure of the integral kernel are decisive factors in ensuring the existence of solutions and their constructive determination.

The research demonstrates that systems of linear integral equations with separable (degenerate) kernels can be reduced to finite-dimensional algebraic systems. This approach allows obtaining analytical expressions for the unknown functions, significantly simplifying the solution process and improving computational accuracy.

The obtained results not only confirm the fundamental principles of integral equation theory but also highlight the importance of method selection in applied modeling problems. In particular, taking into account the structural properties of the integral kernel ensures that the chosen solution method is both theoretically sound and practically effective.

Overall, this study reveals the importance of a systematic and scientifically grounded approach to the analysis of systems of linear integral equations. The results can serve as a theoretical and methodological resource for advanced studies in mathematical analysis and functional analysis, as well as for teaching these subjects in higher educational institutions.

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