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DIFFERENTIAL PROPERTIES OF CURVES

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Abstract: This article comprehensively discusses the differential properties of curves, particularly their parametric representation, arc length, curvature, normal and binormal vectors, Frenet formulas, and their geometric as well as practical interpretations. In addition, the main invariants describing the behavior of curves in space are analyzed.

Keywords: curve, curvature, Frenet–Serret formulas, Frenet frame, tangent vector, normal vector, differential geometry.

Introduction

Differential geometry occupies an important place in the system of modern mathematical sciences and studies geometric objects using the methods of mathematical analysis. This field is aimed at investigating the local and global properties of curves and surfaces and serves as a fundamental theoretical apparatus for determining their structural and analytical characteristics. The theory of curves is considered one of the primary and fundamental branches of differential geometry.

Although the concept of a curve may appear to be one of the simplest geometric objects, its differential properties possess extremely deep and complex meaning. By studying curves, it becomes possible to describe the motion of points in space, the shape of trajectories, geometric deformations, and mathematical models of various physical processes. Therefore, the theory of curves is widely applied not only in pure mathematics, but also in physics, mechanics, astronomy, engineering, and computer graphics.

Historically, the study of curves dates back to the works of ancient Greek scholars. Later, in the seventeenth century, the invention of mathematical analysis made it possible to investigate curves using differential methods. In particular, the establishment of differential calculus by Isaac Newton and Gottfried Wilhelm Leibniz accelerated the development of this field. By the nineteenth century, great mathematicians such as Carl Friedrich Gauss and Bernhard Riemann had established differential geometry as an independent scientific discipline.

In studying the differential properties of curves, primary attention is given to their local characteristics. In other words, quantities describing the behavior of a curve at each point — such as the tangent vector, normal vector, curvature, and torsion — play a significant role. These quantities make it possible to precisely describe the geometric behavior of a curve in space, including how strongly it bends or twists. In particular, the Frenet apparatus provides a complete characterization of curves and represents one of the important achievements of differential geometry.

Today, the differential properties of curves form the theoretical basis of many modern technologies. For example, problems such as constructing smooth curves in computer graphics and animation, optimizing motion trajectories in robotics, and modeling the movement of bodies in aerodynamics and mechanics rely heavily on this theory. Moreover, curves and their properties also play an important role in artificial intelligence and data processing.

The main purpose of this article is to study the differential properties of curves from a theoretical perspective, analyze their principal concepts and mathematical expressions, and

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highlight their practical applications. Furthermore, the article systematically presents such fundamental notions as curvature, torsion, and Frenet formulas.

Main Part

A curve is one of the fundamental objects of differential geometry and is defined as the trajectory of a point moving continuously in space or on a plane. Intuitively, a curve is not a broken or straight figure, but rather a smoothly varying geometric shape. From a mathematical point of view, a curve is defined as follows:

A curve is the set of values of a parametric function defined on a certain interval, that is,

$$\vec{r}(t)=\big(x(t),\, y(t),\, z(t)\big), \qquad t\epsilon[a,b]$$

Here, t - is a parameter (often interpreted as time), while $x(t), y(t), z(t)$ are continuous (usually differentiable) functions.

If the curve lies in a plane, then it can be represented as:

$$\vec{r}(t)=\big(x(t),\, y(t)\big)$$

A curve is called smooth (regular) if

$$\vec{r}'(t)\neq 0$$

that is, if its derivative is not equal to zero. This condition guarantees the existence of a well-defined tangent at every point of the curve.

Curves are classified according to various criteria, such as their dimension, degree of smoothness, whether they are open or closed, and their simplicity properties. There are several main methods for representing curves, each of which is convenient in specific situations. The most general and convenient method is the parametric representation:

$$\vec{r}(t)=\big(x(t),\, y(t),\, z(t)\big)$$

In differential geometry, curvature is one of the fundamental concepts used to measure how much a curve or a surface bends. This concept occupies a central place in differential geometry. The curvature of a curve in the plane or in space is defined as follows:

$$k=\left|\frac{dT}{ds}\right|$$

Here, k - denotes curvature, T - is the unit tangent vector, and s - is the arc length. Curvature indicates how rapidly the tangent vector changes. For a straight line, the curvature is always equal to zero, whereas for a circle the curvature remains constant. For example, when a car makes a sharp turn, the curvature is large; when it moves along a straight road, the curvature is zero.

For surfaces, curvature becomes more complicated. The principal types are:

Gaussian curvature: $K=k_1\cdot k_2$

Mean curvature: $H=\frac{k_1+k_2}{2}$

where k_1, k_2 are the principal curvatures.

Curvature plays an important role in determining geodesic curves and in the classification of surfaces. In the general theory of relativity, the curvature of space-time is interpreted as gravitation. Nowadays, the concept of curvature is developing in such fields as Riemannian geometry and multidimensional spaces, geodesics and variational principles, computer graphics and visualization, as well as artificial intelligence and geometric data analysis. In particular, in Riemannian geometry curvature is generalized through tensors. Therefore, an in-depth study of curvature is of great importance for both mathematical and applied research.

The Frenet–Serret formulas are a fundamental system of equations that completely describe the local geometry of a smooth curve in space. These formulas define the Frenet frame,

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which consists of three mutually perpendicular unit vectors at each point of the curve. Let the curve $r(s)$ be parameterized by arc length s.

Tangent vector: $T(s)=\frac{dr}{ds}$

Normal vector: $N(s)=\frac{T'(s)}{\|T'(s)\|}$

Binormal vector: $B(s)=T(s)\times N(s)$

These three vectors form a mutually orthonormal basis at every point of the curve.

The Frenet formulas indicate the following:

- T' changes only in the normal direction, meaning that bending exists;
- N' changes in two directions, indicating both bending and twisting motion;
- B' is related to torsion, representing spatial twisting.

The Frenet formulas are widely applied in mechanics for trajectory analysis, in robotics for motion planning, in computer graphics for 3D modeling, and in physics for describing particle motion.

Conclusion

The differential properties of curves constitute one of the fundamental branches of differential geometry and provide the possibility for a deep mathematical analysis of the trajectory of a point moving in space. In this article, the parametric representation of curves, arc length, curvature, torsion, the Frenet frame (T, N, B) and the Frenet–Serret formulas were discussed in detail. These concepts make it possible to completely describe the local behavior of a curve: the tangent vector represents the direction of motion, the normal vector expresses the intensity of bending, and the binormal vector describes spatial twisting.

The Frenet apparatus is important not only theoretically but also practically. In robotics, computer graphics and animation, mechanics, aerodynamics, physics, and artificial intelligence, this theory serves as one of the main tools for constructing smooth trajectories, optimizing motion, and modeling the dynamics of objects.

By studying the theory of curves, students and researchers gain a deeper understanding of the analytical properties of geometric objects and develop skills for effectively applying mathematical methods to solve modern scientific and technological problems. In the future, it would be appropriate to investigate this topic further in connection with the differential geometry of surfaces, Riemannian geometry, and computational geometry.

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