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PROBLEM FOR TWO-DIMENSIONAL WAVE EQUATION IN A CIRCLE

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Annotation: This paper investigates the solution of the problem for two-dimensional wave equation in a circle.
Key words: wave equation, circular membrane, radius, boundary conditions, initial conditions, Fourier method, polar coordinate, coefficients, function.

Let us consider the problem of the oscillation of a circular membrane of radius l, fixed along the contour. This problem is reduced to the solution of the wave equation in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}=\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

under the boundary condition

$$u\big|_{r=l}=0 \qquad (2)$$

and initial conditions

$$u\big|_{t=0}=f(r,\varphi), \qquad \frac{\partial u}{\partial t}\bigg|_{t=0}=F(r,\varphi). \qquad (3)$$

From the physical meaning of the problem it is clear that the solution $u(r,\varphi,t)$ must be a single-valued periodic function of φ with a period 2π and remain bounded at all points of the membrane, including the center of the membrane. $r=0$.

Using the Fourier method, we set

$$u(r,\varphi,t)=T(t)\upsilon(r,\varphi). \qquad (4)$$

We get an equation for $T(t)$:

$$T''(t)+a^2\lambda^2T(t)=0,$$

his general solution

$$T(t)=C_1\cos a\lambda t+C_2\sin a\lambda t. \qquad (5)$$

and the following boundary value problem for the function $\upsilon(r,\varphi)$:

$$\frac{\partial^2 \upsilon}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \upsilon}{\partial \varphi^2}+\lambda^2\upsilon=0, \qquad (6)$$

$$\upsilon\big|_{r=l}=0, \qquad (7)$$

$\upsilon|_{r=0}$ = finite value, $\upsilon(r,\varphi)=\upsilon(r,\varphi+2\pi).$ (8)

We will look for a solution to equation (6) in the form

$$\upsilon(r,\varphi)=R(r)\Phi(\varphi). \quad (9)$$

Substituting into equation (6) and separating the variables, we obtain

$$\frac{\Phi''(\varphi)}{\Phi(\varphi)}=-\frac{r^2R''(r)+rR'(r)+\lambda^2r^2R(r)}{R(r)}=-p^2,$$

from where, taking into account (7), (8) and (9), we arrive at two boundary value problems:

$$\Phi''(\varphi)+p^2\Phi(\varphi)=0, \quad (10)$$

$$\Phi(\varphi)=\Phi(\varphi+2\pi), \qquad \Phi'(\varphi)=\Phi'(\varphi+2\pi); \quad (11)$$

$$R''(r)+\frac{1}{r}R'(r)+\left(\lambda^2-\frac{p^2}{r^2}\right)R(r)=0, \quad (12)$$

$$R(l)=0, \qquad |R(0)|<+\infty \quad (13)$$

It is easy to see that non-trivial periodic solutions of problem (10) – (11) exist only under the condition that $p=n$ (an integer) and have the form

$$\Phi_n(\varphi)=A_n\cos n\varphi+B_n\sin n\varphi \quad (n=0,1,2,...).$$

Let us return to equation (12). Its general solution for $p=n$ has the form:

$$R_n(r)=D_nJ_n(\lambda r)+\varepsilon_nY_n(\lambda r).$$

From the second condition (13) it follows that $\varepsilon_n=0$.the first condition gives

$$J_n(\lambda l)=0.$$

By simplification $\lambda l=\mu$, we obtain a transcendental equation for determining μ :

$$J_n(\mu)=0, \quad (14)$$

Which, as we know, has countless positive roots

$$\mu_1^{(n)},\mu_2^{(n)},\mu_3^{(n)},...,$$

which correspond to the values

$$\lambda_{mn}=\frac{\mu_m^{(n)}}{l} \qquad (m=1,2,3,...,\ n=0,1,2,...)$$

and the corresponding solutions to the problem (12) – (13)

$$R_{nm}(r)=J_n\left(\frac{\mu_m^{(n)}r}{l}\right).$$

Returning to the boundary value problem (6) – (7), we obtain that the eigenvalue $\lambda_{mn}^2=\left(\frac{\mu_m^{(n)}}{l}\right)^2$ corresponds to two linearly independent eigenfunctions

$$J_n\left(\frac{\mu_m^{(n)}r}{l}\right)\cos n\varphi, \quad J_n\left(\frac{\mu_m^{(n)}r}{l}\right)\sin n\varphi, \quad (m=1,2,3,...,\ n=0,1,2,...).$$

From the above it follows that it is possible to construct an infinite number of particular solutions of equation (1) that satisfy the boundary condition (2) and have the form:

$$u_{mn}(r,\varphi,t)=\left[A_{nm}\cos\frac{a\mu_m^{(n)}t}{l}+B_{nm}\sin\frac{a\mu_m^{(n)}t}{l}\right]\cos n\varphi+$$

$$+\left[C_{nm}\cos\frac{a\mu_m^{(n)}t}{l}+D_{nm}\sin\frac{a\mu_m^{(n)}t}{l}\right]\sin n\varphi J_n\left(\frac{\mu_m^{(n)}r}{l}\right).$$

To satisfy the initial conditions (3), we will compose a series

$$u(r,\varphi,t)=\sum_{n=0}\sum_{m=1}\left[\left(A_{nm}\cos\frac{a\mu_m^{(n)}}{l}+B_{nm}\sin\frac{a\mu_m^{(n)}}{l}\right)\cos n\varphi+\right.$$

$$\left.+\left(C_{nm}\cos\frac{a\mu_m^{(n)}}{l}+D_{nm}\sin\frac{a\mu_m^{(n)}}{l}\right)\sin n\varphi\right]J_n\left(\frac{\mu_m^{(n)}}{l}\right). \quad (15)$$

Coefficients A_{nm}, B_{nm}, C_{nm}, And D_{nm}, are determined from the initial conditions (3). Indeed, flattening in series (15) $t=0$, we obtain

$$f(r,\varphi)=\sum_{m=1}A_{0m}J_0\left(\frac{\mu_m^{(0)}r}{l}\right)+\sum_{n=1}\left(\sum_{m=1}A_{nm}J_n\left(\frac{\mu_m^{(n)}r}{l}\right)\right)\cos n\varphi+$$

$$+\sum_{n=1}\left(\sum_{m=1}C_{nm}J_n\left(\frac{\mu_m^{(n)}r}{l}\right)\right)\sin n\varphi. \quad (16)$$

This series is the expansion of a periodic function $f(r,\varphi)$ into a Fourier series in the interval $(0,2\pi)$ and, therefore, the factors here at $\cos n\varphi$ and $\sin n\varphi$ must be Fourier coefficients; in other words, the following equalities must hold:

$$\frac{1}{2\pi}\int_0^{2\pi}f(r,\varphi)\cos n\varphi d\varphi=\sum_{m=1}A_{nm}J_n\left(\frac{\mu_m^{(n)}r}{l}\right), \quad (17)$$

$$\frac{1}{\pi}\int_0^{2\pi}f(r,\varphi)\cos n\varphi d\varphi=\sum_{m=1}A_{nm}J_n\left(\frac{\mu_m^{(n)}r}{l}\right), \quad (18)$$

$$\frac{1}{\pi}\int_0^{2\pi}f(r,\varphi)\sin n\varphi d\varphi=\sum_{m=1}C_{nm}J_n\left(\frac{\mu_m^{(n)}r}{l}\right). \quad (19)$$

Considering these equalities, we see that they represent expansions of an arbitrary function $\Phi(r)$ into a series of Bessel functions:

$$\Phi(r)=\sum_{m=1}a_mJ_n\left(\frac{\mu_m^{(n)}r}{l}\right).$$

The coefficients a_m are determined by the formula

$$a_m=\frac{2}{l^2J_{n+1}^2\left(\mu_m^{(n)}\right)}\int_0^l r\Phi(r)J_n\left(\frac{\mu_m^{(n)}}{l}\right)dr.$$

Taking this formula into account, we can easily see that

$$A_{0m}=\frac{2}{\pi l^2J_1^2\left(\mu_m^{(0)}\right)}\int_0^l\int_0^{2\pi}f(r,\varphi)J_0\left(\frac{\mu_m^{(0)}r}{l}\right)rdrd\varphi, \quad (20)$$

$$A_{nm}=\frac{2}{\pi l^2 J_{n+1}^2\left(\mu_m^{(n)}\right)}\int_0^l\int_0^{2\pi} f(r,\varphi)J_n\left(\frac{\mu_m^{(n)}r}{l}\right)\cos n\varphi r dr d\varphi, \qquad (21)$$

$$C_{nm}=\frac{2}{\pi l^2 J_{n+1}^2\left(\mu_m^{(n)}\right)}\int_0^l\int_0^{2\pi} f(r,\varphi)J_n\left(\frac{\mu_m^{(n)}r}{l}\right)\sin n\varphi r dr d\varphi. \qquad (22)$$

Reasoning in a similar way, we will also determine the coefficients $B_{0m},\ B_{nm},\ D_{nm}$ $-$we only need to replace in formulas (20), (21) and (22) $f(r,\varphi)$ with $F(r,\varphi)$ and divide the corresponding expressions by $\frac{a\mu_m^{(n)}}{l}$. Thus, all the coefficients in the expansion (15) are determined, and we can rewrite the solution we found for problem (1) – (3) in the form:

$$u(r,\varphi,t)=\sum_{n=0}\sum_{m=1} M_{nm}J_n\left(\frac{\mu_m^{(n)}r}{l}\right)\sin(n\varphi+\psi_{nm})\sin\left(\frac{\mu_m^{(n)}at}{l}+\nu_{nm}\right), \qquad (23)$$

where are the constants M_{nm}, ψ_{nm} and ν_{nm} are obviously connected with constants A_{nm}, B_{nm}, C_{nm} And D_{nm}.

In the case of radial oscillations of a circular membrane, the initial functions depend only on r:

$$u\big|_{t=0}=f(r), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=F(r). \qquad (24)$$

Then from formulas (20) (21), (22) and similar ones it follows that

$$A_{0m}=A_{0m}=\frac{2}{l^2J_1^2\left(\mu_m^{(0)}\right)}\int_0^l rf(r,\varphi)J_0\left(\frac{\mu_m^{(0)}r}{l}\right)dr,$$

$$B_{0m}=A_{0m}=\frac{2}{alJ_1^2\left(\mu_m^{(0)}\right)}\int_0^l rF(r,\varphi)J_0\left(\frac{\mu_m^{(0)}r}{l}\right)dr,$$

and at $n>0$ the coefficients A_{nm}, B_{nm}, C_{nm} and D_{nm} are equal to zero.

Row (15) is reduced to row

$$u(r,t)=\sum_{m=1}\left(A_{0m}\cos\frac{a\mu_m^{(0)}t}{l}+B_{0m}\sin\frac{a\mu_m^{(0)}t}{l}\right)J_0\left(\frac{\mu_m^{(0)}r}{l}\right), \qquad (25)$$

where are $\mu_m^{(0)}$ $-$the positive roots of the equation $J_0(\mu)=0$.

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